Math 3236 Statistical Theory

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Human decisions. Economics.

N individuals that take

a decision n times each.

Nn Bernoulli r.v.

They are all independent

p is the probability of $\mathcal{L}$.

$$\frac{\# \mathcal{L}}{Nn} = \hat{p}$$

Heterogeneity.

$p(a)$ a is individual

$p = D + u(a)$ not nice.

Suppose that each individual a

has a threshold

$D + u(a)$

When he must decide, he extract a noise ε and

if $\varepsilon > D + u(a) \Rightarrow 0$

$\varepsilon < D + u(a) \Rightarrow 1$

ε has $N(0,1)$

$$p(a) = \Phi(D + u(a))$$

$$(1 - p(a)) = \Phi(-D - u(a))$$

$$\Phi(-x) = 1 - \Phi(x)$$

$\sigma(i, a)$ is the i-th answer from the a individual

$$L(D, u(a)) = \prod_{i=1}^{n} \prod_{a=1}^{N} \Phi\big(\sigma(i,a)(D + u(a))\big)$$

Where $\sigma(i,a) = 1$ if answer is 1

$\sigma(i,a) = -1$ if answer is 0

In general n is much smaller Then N. Not enough data To estimate all $u(a)$.

Assumption

$u(a)$ is distributed as a $N(0, \sigma^2)$

$$L(a) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\big(\sigma(i,a)(D + u)\big) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \, du$$

$$L(\Sigma) = \prod_{a=1}^{N} L(a)$$

I can use $L(\Sigma)$ ha a

likelihood function, Maximizing it D, σ^2.

Chapter 8

Sampling distribution for estimators.

$X_1 \quad \dots \quad X_N$

$r(\underline{X})$ LLN C.L.T.

What is the p.d.f. of $r(\underline{X})$?

Examples:

X_i are normal with μ unknown, σ^2 Known

$\bar{X} = \frac{1}{N}\sum_i X_i$ is an estimator for μ. Unbaised.

How far is $\bar{X}$ from μ?

In the case of Normal r.v.

$$\bar{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{N}\right) \quad \text{exact}$$

$$\mathbb{P}\left(|\bar{X} - \mu| \geq 0.1 \right) =$$

$$\mathbb{P}\left(-0.1 \leq X - \mu \leq 0.1\right) =$$

$$\mathbb{P}\left(-\frac{0.1}{\sigma/\sqrt{N}} \leq \frac{X-\mu}{\sigma/\sqrt{N}} \leq \frac{0.1}{\sigma/\sqrt{N}}\right).$$

$\frac{X-\mu}{\sigma/\sqrt{N}}$ is $\mathcal{N}(0,1)$

$$= \Phi\left(\frac{0.1}{\sigma/\sqrt{N}}\right) - \Phi\left(-\frac{0.1}{\sigma/\sqrt{N}}\right) =$$

$$= 1 - 2\Phi\left(-\frac{0.1}{\sigma/\sqrt{N}}\right) =$$

$$= 1 - 2\Phi\left(-\frac{0.1\sqrt{N}}{\sigma}\right)$$

If on the other side we know μ but σ is unknown.

$$X_i \quad \mathcal{N}(\mu, \sigma^2)$$

$$L(\sigma, X) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\sum_i \left[\frac{(x_i-\mu)^2}{2\sigma^2}\right]}$$

$$\tau = \frac{1}{\sigma} \qquad = \left(\frac{\tau}{\sqrt{2\pi}}\right)^N e^{-\sum_i \frac{\tau^2}{2}(x_i-\mu)^2}$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^N \left(N\tau^{N-1} e^{-\sum_i \frac{\tau^2}{2}(x_i-\mu)^2} - \tau^N \tau \sum_i (x_i-\mu)^2 e^{-\frac{\tau^2}{2}\sum_i (x_i-\mu)^2} \right)$$

$$= \partial_\tau L(\tau)$$

$$\partial_\tau L(\tau) = 0 \quad \Rightarrow$$

$$N - \tau^2 \sum_i (x_i-\mu)^2 = 0$$

$$\tau^2 = \frac{N}{\sum_i (x_i-\mu)^2}$$

$$\hat{\sigma}^2(\underline{X}) = \frac{1}{N} \sum_i (X_i - \mu)^2$$

Un buised ?

$$\mathbb{E}(\hat{\sigma}^2(X)) = \frac{1}{N} \sum_{i,j} \mathbb{E}((X_i - \mu)(X_j - \mu))$$

$$\frac{1}{N} \sum_{i \neq j} \mathrm{Cov}(X_i, X_j) +$$

$$\frac{1}{N} \sum_i \mathrm{Var}(X_i) = \sigma^2$$

$$\hat{\sigma}(\underline{X}) = \sqrt{\hat{\sigma}^2(\underline{X})}$$

$\hat{\sigma}(X)$ is not unbiased !!

$$\sqrt{\mathbb{E}(\hat{\sigma}^2(\underline{X}))} = \sigma$$

$$\cancel{=} \qquad \neq$$

$$\mathbb{E}\left(\sqrt{\hat{\sigma}^2(\underline{X})}\right)$$

What is the distribution of

$$\hat{\sigma}^2(\underline{X}) = \frac{1}{N} \sum_i (X_i - \mu)^2$$

$$Y_i = \left(\frac{X_i - \mu}{\sigma}\right)^2$$

p.d.f. of

$$\frac{X_i - \mu}{\sigma} = z$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$f_Y(y) = \frac{2}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}$$

$$f_Z(z)\,dz \rightarrow \frac{1}{\sqrt{2\pi}} \frac{y}{2}^{-\frac{1}{2}} e^{-y/2}\,dy$$

$$z^2 = y \qquad 2z\,dz = dy$$

$$dz = \frac{1}{2z}\,dy = \frac{1}{2\sqrt{y}}\,dy$$

$$(-Z)^2 = Y$$

$y = 1 \qquad z = 1 \qquad z = -1$

$$f_Z(z) = \frac{1}{\sqrt{\pi}} \frac{y^{-1/2}}{\sqrt{2}} e^{-\frac{y}{2}}$$

χ^2 with 1 degree of freedom.

This is $\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$

$$f_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \beta^{\alpha} e^{-\beta x}$$

$\beta = \frac{1}{2} \qquad \alpha = \frac{1}{2} \qquad \left(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right)$

$$N \frac{\hat{\sigma}^2(X)}{\sigma^2} = \sum_i \left(\frac{X_i - \mu}{\sigma}\right)^2 =$$

$$\sum_i \chi_i$$

if X is $\Gamma(\alpha, \beta)$ and

$$Y \sim \Gamma(\alpha', \beta)$$

$$\Downarrow$$

$$X + Y \text{ is } \Gamma(\alpha + \alpha', \beta)$$

$$\sum_i X_i \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

$$\sim \frac{1}{\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2} - 1} \left(\frac{1}{2}\right)^{n/2} e^{-y/2}$$

χ^2 distribution with n d.o.f.

$$\boxed{\frac{N \hat{\sigma}^2(\underline{X})}{\sigma^2}} \sim \chi^2_n$$

$\searrow$ pivotal quantity.

$$\mathbb{P}\left(\left| \frac{\hat{\sigma}^2(\underline{X})}{\sigma^2} - 1 \right| > 0.1 \right)$$

What is the prob that in my estimation I make an error of more than 10%.

$$\mathbb{P}\left(\left| \frac{\hat{\sigma}^2(X)}{\sigma^2} - 1 \right| > 0.1 \right) =$$

$$= \mathbb{P}\left(0.9 < \frac{\hat{\sigma}^2(X)}{\sigma^2} < 1.1 \right)$$

$$= \mathbb{P}\left(N \cdot 0.9 \leq \underbrace{N \frac{\hat{\sigma}^2(X)}{\sigma^2}}_{\chi^2_n} \leq 1.1 \cdot N \right)$$